Section 16.1 – 16.2 WKST SOLUTIONS

Section 16.1

1. Let $R = [2,4] \times [-2,3]$. Then, $\iint_R 7\, dA = 7 \iint_R dA$

$$= 7 \cdot \text{area}(R)$$

$$= 7 \cdot 2 \cdot 5$$

$$= \boxed{70.}$$

2. Evaluate $\displaystyle\int_{-1}^{1}\int_{0}^{\pi} x^2 \sin y \; dy dx$.

$$\int_{-1}^{1}\int_{0}^{\pi} x^2 \sin y \, dy \, dx = \int_{-1}^{1} x^2 \left(\int_{0}^{\pi} \sin y \, dy \right) dx$$

$$= \int_{-1}^{1} x^2 \left(-\cos(\pi) + \cos(0) \right) dx = \int_{-1}^{1} 2x^2 \, dx$$

$$= \frac{2x^3}{3} \Big|_{-1}^{1} = \frac{2}{3} + \frac{2}{3} = \boxed{\frac{4}{3}}$$

3. Evaluate $\int_0^1\int_0^1 y\sqrt{1+xy}\,dy\,dx$. Hint: change the order of integration.

$$\int_0^1\int_0^1 y\sqrt{1+xy}\,dx\,dy = \int_0^1\left(\int_1^{1+y} \frac{y\,u^{1/2}}{y}\,du\right)dy$$

Let $u = 1+xy$

$du = y\,dx$

$$= \int_0^1 \left(\frac{2u^{3/2}}{3}\Big|_1^{1+y}\right) dy$$

$$= \frac{2}{3}\int_0^1 (1+y)^{3/2} - 1\,dy = \frac{2}{3}\left[\frac{2(1+y)^{5/2}}{5} - y\Big|_0^1\right]$$

$$= \frac{2}{3}\left[\frac{2}{5}\cdot 2^{5/2} - 1 - \frac{2}{5}\right]$$

$$= \frac{2^2\cdot 2^2\sqrt{2}}{15} - \frac{2}{3} - \frac{4}{15}$$

$$= \boxed{\frac{16\sqrt{2}}{15} - \frac{14}{15}}$$

4. Let $f(x,y) = mxy^2$ where m is a constant. Find a value of m such that $\iint_R f(x,y)\,dA = 1$ where $R = [0,1] \times [0,2]$.

$$\iint_R f(x,y)\,dA = \int_0^2 \int_0^1 mxy^2 \,dx\,dy = m \int_0^2 \left(\int_0^1 xy^2 \,dx \right) dy$$

$$= m \int_0^2 \left(\frac{x^2}{2} \cdot y^2 \Big|_0^1 \right) dy = \frac{m}{2} \int_0^2 y^2 \,dy$$

$$= \frac{m y^3}{6} \Big|_0^2$$

$$= \frac{m\,8}{6}$$

$$= \frac{m4}{3}.$$

$\boxed{\text{So, } m = \frac{3}{4}} \Rightarrow \iint_R f(x,y)\,dA = 1.$

Section 16.2 Additional Exercises

1. Compute the double integral $\iint_R x^2 y \, dA$

where R is the region described by $1 \le x \le 3$
$x \le y \le 2x+1$.

$$\iint_R x^2 y \, dA = \int_1^3 \int_x^{2x+1} x^2 y \quad dy \, dx$$

$$= \int_1^3 \left(\int_x^{2x+1} x^2 y \quad dy \right) dx$$

$$= \int_1^3 \left(\frac{x^2 y^2}{2} \Big|_x^{2x+1} \right) dx$$

$$= \int_1^3 \frac{x^2(2x+1)^2}{2} - \frac{x^4}{2} \quad dx$$

$$= \frac{1}{2} \int_1^3 4x^4 + 4x^3 + x^2 - x^4 \, dx \longrightarrow$$

$$= \frac{1}{2}\int_1^3 3x^4 + 4x^3 + x^2 \, dx$$

$$= \frac{1}{2}\left[\frac{3x^5}{5} + \frac{4x^4}{4} + \frac{x^3}{3} \Big|_1^3 \right]$$

$$= \frac{1}{2}\left[\frac{3^6}{5} + 3^4 + 3^2 - \frac{3}{5} - 1 - \frac{1}{3} \right]$$

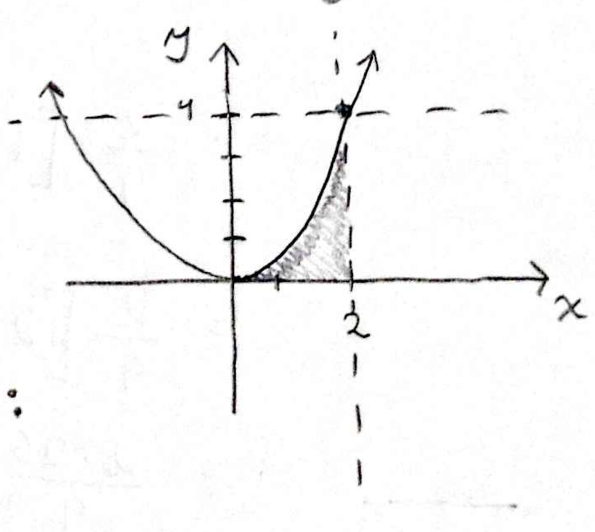
$$= \boxed{\frac{1754}{15} \approx 116.93}$$

2. Sketch the domain of integration. Then change the order of integration and evaluate. Explain the simplification achieved by changing the order.

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1} \; dx \, dy$$

$\sqrt{y} \leq x \leq 2$

$0 \leq y \leq 4$



The region is described as horizontally simple. To change the order of integration we write it as vertically simple:

$0 \leq x \leq 2 \qquad 0 \leq y \leq x^2$

↳

So, changing the order of integration gives

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\, dx\, dy = \int_0^2 \int_0^{x^2} \sqrt{x^3+1}\, dy\, dx$$

$$= \int_0^2 \left(y\sqrt{x^3+1} \Big|_0^{x^2} \right) dx$$

$$= \int_0^2 x^2 \sqrt{x^3+1}\, dx$$

Let $u = x^3+1$, $du = 3x^2\, dx$

$$= \frac{1}{3} \int_1^9 u^{1/2}\, du$$

$$= \frac{1}{3} \left[\frac{2u^{3/2}}{3} \Big|_1^9 \right]$$

$$= \frac{2}{9} \left[(9)^{3/2} - 1 \right]$$

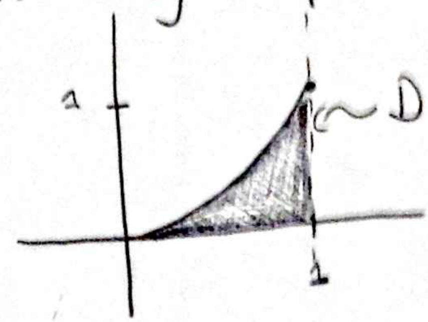
$$= \frac{2}{9} [27 - 1]$$

$$= \frac{2 \cdot 26}{9} = \boxed{\frac{52}{9}}$$

Calculate the average height above the x-axis of a point in the region $0 \le x \le 1$, $0 \le y \le x^2$.

The height above the x-axis is given by $f(x,y) = y$.

vertically simple.



$$\text{Average} = \frac{1}{\text{Area}(D)} \iint_D y \, dA.$$

Step 1)
$$\iint_D y\,dA = \int_0^1 \int_0^{x^2} y \, dy\, dx = \int_0^1 \left(\frac{y^2}{2}\Big|_0^{x^2} \right) dx$$

$$= \int_0^1 \frac{x^4}{2} \, dx$$

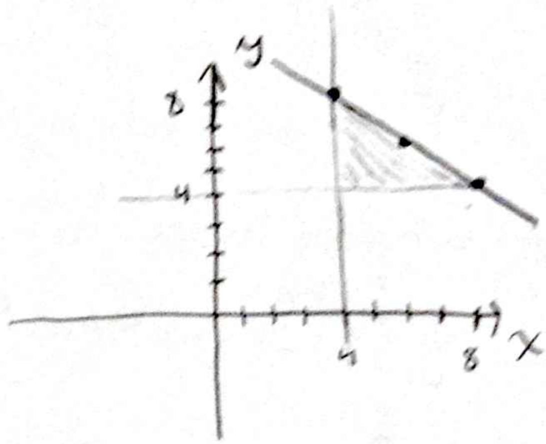
$$= \frac{x^5}{10}\Big|_0^1 = \frac{1}{10}.$$

Step 2)
$$\text{Area}(D) = \int_0^1 \int_0^{x^2} dy\,dx = \int_0^1 x^2 \, dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}.$$

$$\boxed{\text{So, average is } \frac{3}{10}}$$

Extra Problems

1. Sketch the domain D defined by $x+y \leq 12$, $x \geq 4$, $y \geq 4$ and compute $\iint_D e^{x+y}\,dA$.



This region can be expressed as vertically simple:

$4 \leq x \leq 8$

$4 \leq y \leq 12 - x$.

So, $$\iint_D e^{x+y}\,dA = \int_4^8 \int_4^{12-x} e^x \cdot e^y \,dy\,dx$$

$$= \int_4^8 \left(e^x e^y \Big|_4^{12-x} \right) dx$$

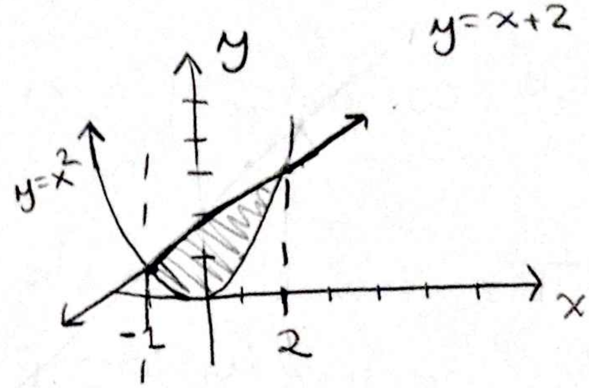
$$= \int_4^8 e^x e^{12-x} - e^x e^4 \,dx$$

$$= \int_4^8 e^{12} - e^{x+4} \,dx = e^{12}x - e^{x+4} \Big|_4^8$$

$$= 4e^{12} - e^{12} + e^8$$

$$= \boxed{3e^{12} + e^8}$$

2. Integrate $f(x,y) = x$ over the region bounded by $y = x^2$ and $y = x+2$.



First, we find the points where $y = x^2$ and $y = x+2$ intersect.

$x^2 - x - 2 = 0 \Rightarrow$

$(x-2)(x+1) = 0 \Rightarrow$

$x = 2,\ x = -1$.

So, the region is vertically simple, and we can express it as:

$$-1 \le x \le 1$$
$$x^2 \le y \le x+2.$$

So, $$\iint_D x \, dA = \int_{-1}^{2}\int_{x^2}^{x+2} x \, dy \, dx = \int_{-1}^{2}\left(xy \Big|_{x^2}^{x+2} \right) dx$$

$$= \int_{-1}^{2} x^2 + 2x - x^3 \, dx = \frac{x^3}{3} + x^2 - \frac{x^4}{4} \Big|_{-1}^{2}$$

$$\frac{8}{3} + 4 - 4 + \frac{1}{3} - 1 + \frac{1}{4} = 3 - \frac{3}{4} = \frac{9}{4} = \boxed{2\frac{1}{4}}$$